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Large-time existence of compressible viscous and heat-conductive surface waves

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1 Introduction and theorem.

In this communication we are concerned with free boundary problem for compressible viscous isotropic Newtonian fluid which is formulated as follows: Find the domain $\Omega_t \subset \boldsymbol{R}^3$ occupied by the fluid at the moment $t > 0$ together with the density $\rho(x,t)$, velocity vector field $v(x,t) = (v_1, v_2, v_3)$ and with the absolute temperature $\theta(x,t)$ satisfying the system of Navier-Stokes equations

$$
(1.1) \quad \begin{cases} \dfrac{D\rho}{Dt} + \rho(\nabla\cdot v) = 0, \quad \rho\dfrac{Dv}{Dt} = \nabla\cdot\boldsymbol{P} - \rho g e_3, \\ \rho c_V \dfrac{D\theta}{Dt} + \theta p_\theta(\nabla\cdot v) = \nabla\cdot(\kappa\nabla\theta) + \Psi, \\ x \in \Omega_t \equiv \{x' = (x_1,x_2) \in \boldsymbol{R}^2, -b(x') < x_3 < F(x',t)\},\ t > 0 \end{cases}
$$

and the initial and boundary conditions

$$
(1.2) \quad \begin{cases} (\rho, v, \theta)|_{t=0} = (\rho_0, v_0, \theta_0), \quad x \in \Omega_0, \\ \boldsymbol{P}n = -p_e n + \sigma H n, \quad \kappa\nabla\theta\cdot n = \kappa_e(\theta_e - \theta), \\ x \in \Gamma_t \equiv \{x' \in \boldsymbol{R}^2,\ x_3 = F(x',t)\}, \quad t > 0, \\ v = 0, \quad \theta = \theta_a, \quad x \in \Sigma \equiv \{x' \in \boldsymbol{R}^2,\ x_3 = -b(x')\}, \quad t > 0, \\ \dfrac{D}{Dt}(x_3 - F) = 0, \quad x \in \Gamma_t,\ t > 0, \quad F|_{t=0} = F_0(x'), \quad x' \in \boldsymbol{R}^2. \end{cases}
$$

Here, $\nabla = (\frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2}, \frac{\partial}{\partial x_3})$, $\frac{D}{Dt} = \frac{\partial}{\partial t} + (v \cdot \nabla)$ is the material derivative, $\boldsymbol{P} = (-p + \mu'(\nabla \cdot v))\boldsymbol{I} + 2\mu \boldsymbol{D}(v) \equiv -p\boldsymbol{I} + \boldsymbol{V}$ is the stress tensor, $\boldsymbol{I}$ is the 3×3 unit matrix, $\boldsymbol{D}(v)$ is the velocity deformation tensor with the elements $D_{ij} = \frac{1}{2}(\frac{\partial v_i}{\partial x_j} + \frac{\partial v_j}{\partial x_i})$, $\Psi = \mu'(\nabla \cdot v)^2 + 2\mu \boldsymbol{D}(v) : \boldsymbol{D}(v)$ is the dissipation function, $p = p(\rho, \theta)$ is the pressure, $(\mu, \mu', \kappa, c_V)(\rho, \theta)$ are, respectively, coefficient of viscosity, second coefficient of viscosity, coefficient of heat conductivity, heat capacity at constant volume, which are all assumed to be known smooth functions of (ρ, θ) satisfying $\mu, \kappa, c_V > 0$, $2\mu + 3\mu' \geq 0$, $p_\rho, p_\theta > 0, (g, \sigma, p_e, \kappa_e)$ are, respectively, acceleration of gravity, coefficient of surface tention, atmospheric pressure, coefficient of outer heat conductivity, which are all assumed to be positive constants, $e_3 = {}^t(0,0,1)$, $n = \frac{1}{\sqrt{1+|\nabla' F|^2}} {}^t(-\nabla_1 F, -\nabla_2 F, 1)$ is the exterior unit normal vector to Γ_t, $\nabla' = (\nabla_1, \nabla_2) = (\frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2})$ and $H = \nabla' \cdot (\frac{\nabla' F}{\sqrt{1+|\nabla' F|^2}})$ is the twice mean curvature of Γ_t.

We seek a solution near the equilibrium rest state $(\rho, v, \theta, F) = (\bar{\rho}, 0, \bar{\theta}, 0)$, where $\bar{\theta}$ is any positive constant and $\bar{\rho} = \bar{\rho}(x_3)$ is determined by

$$(1.3) \qquad \int_{\bar{\rho}(0)}^{\bar{\rho}(x_3)} \frac{p_\rho(\eta, \bar{\theta})}{\eta} d\eta + g x_3 = 0, \qquad p(\bar{\rho}(0), \bar{\theta}) = p_e.$$

We rewrite the problem (1.1),(1.2) by the change of unknown functions $(\rho, v, \theta, F) \to (\rho + \bar{\rho}, v, \theta + \bar{\theta}, F)$ using (1.3) as follows:

$$(1.4) \begin{cases} \dfrac{D}{Dt}(\rho + \bar{\rho}) + (\rho + \bar{\rho})(\nabla \cdot v) = 0, \\ (\rho + \bar{\rho})\dfrac{Dv}{Dt} = \nabla \cdot \boldsymbol{V} - p_\rho \nabla \rho - p_\theta \nabla \theta + (\dfrac{\bar{\rho}}{\bar{p}_\rho}(p_\rho - \bar{p}_\rho) - \rho) g e_3, \\ (\rho + \bar{\rho}) c_V \dfrac{D\theta}{Dt} + (\theta + \bar{\theta}) p_\theta (\nabla \cdot v) = \nabla \cdot (\kappa \nabla \theta) + \Psi, \ x \in \Omega_t, t > 0, \end{cases}$$

$$
(1.5)\qquad \begin{cases}
(\rho, v, \theta)|_{t=0} = (\rho_0, v_0, \theta_0)(x), \quad x \in \Omega_0, \\
2\mu\Pi \boldsymbol{D}(v) = 0, \quad -(p - p_e) + \boldsymbol{V} n\cdot n = \sigma H, \\
\kappa \nabla\theta\cdot n = \kappa_e(\theta_e - \theta), \quad x \in \Gamma_t, t > 0, \\
v = 0, \ \theta = \theta_a, \quad x \in \Sigma, t > 0, \\
F_t + v_1\nabla_1 F + v_2 \nabla_2 F - v_3 = 0, \quad x \in \Gamma_t, t > 0, \\
F|_{t=0} = F_0(x'), \quad x' \in \boldsymbol{R}^2,
\end{cases}
$$

where $p = p(\rho + \bar{\rho}, \theta + \bar{\theta})$, $\bar{p}_\rho = p_\rho(\bar{\rho}, \bar{\theta})$ *etc.*, and $\Pi\varphi = \varphi - n(n \cdot \varphi)$.

We consider the problem (1.4),(1.5) in S.L.Sobolev-L.N.Slobodetskiĭ spaces. Let G be a domain in $\boldsymbol{R}^n$ and $l > 0$ be not an integer. By $W_2^l(G)$ we mean the space of functions $u(x)$ ($x \in G$) equipped with the norm

$$
\begin{aligned}
\|u\|^2_{W_2^l(G)} &= \sum_{|j|<l} \|D^j u\|^2_{L_2(G)} + \\
&\quad + \sum_{|j|=[l]} \int_G \int_G \frac{|D^j u(x) - D^j u(y)|^2}{|x-y|^{n+2(l-[l])}} dx dy.
\end{aligned}
$$

Now we define an anisotropic spaces $W_2^{l,l/2}(Q_T)$ ($Q_T = \Omega \times (0,T)$) consisting of functions $u(x,t)$ ($(x,t) \in Q_T$) by $W_2^{l,l/2}(Q_T) = L_2(0,T; W_2^l(\Omega)) \cap L_2(\Omega; W_2^{l/2}(0,T))$ and introduce in this space the norm

$$
\|u\|^2_{W_2^{l,l/2}(Q_T)} = \int_0^t \|u(\cdot, t)\|^2_{W_2^l(\Omega)} dt + \int_\Omega \|u(x, \cdot)\|^2_{W_2^{l/2}(0,T)} dx.
$$

The same notation will be used for the spaces of vector fields, the norms of a vector supposed to be equal to the sum of all its components.

Let us first state local solvability of the problem (1.4),(1.5). Transforming the problem to the initial domain Ω_0 by the relation

$$
(1.6)\qquad x = \xi + \int_0^t \hat{v}(\xi, \tau) d\tau \equiv x(\xi, t),
$$

where $\hat{v}(\xi, t)$ is the velocity vector field in Lagrangean coordinate system, we have

Theorem 1.1 (local existence) *Let* $b \in W_2^{5/2+l}(\boldsymbol{R}^2)$ *with* $l \in (1/2,1)$. *For arbitrary* $\rho_0, v_0, \theta_0 \in W_2^{2+l}(\Omega_0), \rho_0+\bar{\rho}, \theta_0+\bar{\theta}>0, F_0 \in W_2^{7/2+l}(\boldsymbol{R}^2), \theta_e \in W_2^{4+l,2+l/2}(\boldsymbol{R}_T^3), \theta_a \in W_2^{5/2+l,5/4+l/2}(\Sigma_T), \theta_e+\bar{\theta}, \theta_a+\bar{\theta}>0$ *satisfying natural compatibility conditions, which we omit them here, the problem (1.4), (1.5) in Lagrangean coordinate system has the unique solution* $(\hat{\rho}, \hat{v}, \hat{\theta})(\xi,t)$ *defined on* $Q_{T_1} \equiv \Omega_0 \times (0,T_1)$ *for some* $T_1 \in (0,T)$ *such that* $\hat{\rho} \in W_2^{2+l,1+l/2}(Q_{T_1})$, $\hat{v}, \hat{\theta} \in W_2^{3+l,3/2+l/2}(Q_{T_1})$ *and*

$$\begin{aligned} \hat{E}^{3+l}(Q_{T_1}) &\equiv \|\hat{\rho}\|_{W_2^{2+l,1+l/2}(Q_{T_1})} + \|(\hat{v},\hat{\theta})\|_{W_2^{3+l,3/2+l/2}(Q_{T_1})} \le \\ &\le c_1 \Big(\|(\rho_0,v_0,\theta_0)\|_{W_2^{2+l}(\Omega_0)} + \|F_0\|_{W_2^{7/2+l}(R^2)} + \\ &\quad + \|\theta_e\|_{W_2^{4+l,2+l/2}(R_T^3)} + \|\theta_a\|_{W_2^{3/2+l,3/4+l/2}(\Sigma_T)} \Big) \equiv c_1 E_{0,T}. \end{aligned} \tag{1.7}$$

The number T_1 *increases unboundedly as* $E_{0,T}$ *tends to zero. Moreover, the solution possesses some additional regularity with respect to* $t \ge t_1$:

$$\sup_{t_1<t<T_1} \Big(\|\hat{\rho}\|_{W_2^{2+l}(\Omega_0)} + \|(\hat{v},\hat{\theta})\|_{W_2^{3+l}(\Omega_0)} \Big) \le c_2 (E_{0,T} + \hat{E}^{3+l}(Q_{T_1})). \tag{1.8}$$

with arbitrary positive $t_1 \le T_1$.

The proof of Theorem 1.1 can be carried out in the same way as in [5,8].

The following is our main theorem.

Theorem 1.2 (global existence) *Under the assumptions of theorem 1.1, if* $E_0 \equiv E_{0,\infty} \le \varepsilon$ *with sufficiently small number* ε, *then the problem (1.4),(1.5) has the unique solution* (ρ, v, θ, F) *for all* $t>0$ *satisfying*

$$\sup_{t \ge t_1} \Big(\|\rho\|_{W_2^{2+l}(\Omega_t)} + \|(v,\theta)\|_{W_2^{3+l}(\Omega_t)} + \|F\|_{W_2^{7/2+l}(R^2)} \Big) \le c_3 E_0 \tag{1.9}$$

with each $t_1 > 0$.

Similar result for barotropic fluid bounded only by a free surface was established in [6].

2 Proof of theorem 1.2.

Theorem 1.2 is proved by combination of the local existence theorem and the a priori estimate. To state the a priori estimate, it is convienient to make use of the coordinate transformation mapping from Ω_t onto the equilibrium domain $\bar{\Omega} \equiv \{y' \in \boldsymbol{R}^2, -b(y') < y_3 < 0\}$ defined by

$$(x_1, x_2, x_3) = (y_1, y_2, \tilde{F} + y_3(1 + \frac{\tilde{F}}{b})) \equiv x(y, t), \tag{2.1}$$

where $\tilde{F}$ is the extension of F to $\bar{\Omega} \times \boldsymbol{R}_+$ (see [1]). Let us put $\tilde{f}(y,t) = f(x(y,t),t)$ and

$$\begin{aligned} \tilde{E}^{3+l}(\bar{Q}_T) \equiv\ & \|\tilde{\rho}\|_{W_2^{2+l,1+l/2}(\bar{Q}_T)} + \|(\tilde{v},\tilde{\theta})\|_{W_2^{3+l,3/2+l/2}(\bar{Q}_T)} + \\ & + \|F\|_{W_2^{7/2+l,7/4+l/2}(R_T^2)}, \quad \bar{Q}_T = \bar{\Omega} \times (0,T). \end{aligned}$$

Theorem 2.1 (a priori estimate) *Let (ρ, v, θ, F) be the solution of (1.4),(1.5) defined on $0 < t < T$. If $E_{0,T} < \varepsilon_1$ and $\tilde{E}^{3+l}(\bar{Q}_T) < \delta_1$ with sufficiently small ε_1, δ_1, then the following a priori estimate holds:*

$$\tilde{E}^{3+l}(\bar{Q}_T) \leq c_4 E_{0,T}. \tag{2.2}$$

Proof of Theorem 1.2. Let E_0 be so small that the problem (1.4,),(1.5) is solvable on the interval $(0,1)$. Such a solution satisfies inequalities (1.7),(1.8) for $T_1 = 1$. Furthermore, (2.2) with $T = 1$ is valid provided that $E_0 < \varepsilon_1$ and $c_1 E_0 < \delta_1$. Combining these inequalities, we find that $E_1 \leq c_5 E_0$ (E_1 is the norms of the data at $t = 1$). Introducing new Lagrangean coordinate system $\xi \in \Omega_1$ and again applying Theorem 1.1, we can establish the solvability of the problem for $t \in (1,2)$ provided that E_0 is sufficiently small. Repeating this process infinitely many times, we arrive at the assertion of the theorem. ∎

3 a priori estimate.

First we rewrite the system (1.4),(1.5) so that all the nonlinear terms appear in the right hand side of equations and next make transformation to the equilibrium rest domain $\bar{\Omega}$ and linearize it again. Then we finally obtain

$$(3.1)\qquad \begin{cases} \tilde{\rho}_t + \bar{\rho}(\nabla\cdot\tilde{v}) + (\tilde{v}\cdot\nabla)\bar{\rho} = f^1, \\ \bar{\rho}\tilde{v}_t - \nabla\cdot\tilde{\boldsymbol{V}} + \bar{p}_\rho\nabla\tilde{\rho} + \bar{p}_\theta\nabla\tilde{\theta} - \\ \qquad -\Big(\dfrac{\bar{\rho}}{\bar{p}_\rho}(dp_\rho)_{(\bar{\rho},\bar{\theta})}(\tilde{\rho},\tilde{\theta}) - \tilde{\rho}\Big)ge_3 = f^2, \\ \bar{\rho}\bar{c}_V\tilde{\theta}_t - \nabla\cdot(\bar{\kappa}\nabla\tilde{\theta}) + \bar{\theta}\bar{p}_\theta(\nabla\cdot\tilde{v}) = f^3, \quad in\ \bar{Q}_T, \end{cases}$$

$$(3.2)\qquad \begin{cases} (\tilde{\rho},\tilde{v},\tilde{\theta})|_{t=0} = (\tilde{\rho}_0,\tilde{v}_0,\tilde{\theta}_0)(y), \quad on\ \bar{\Omega}, \\ \bar{\mu}\left(\dfrac{\partial\tilde{v}_k}{\partial y_3} + \dfrac{\partial\tilde{v}_3}{\partial y_k}\right)\Big|_{y_3=0} = f^{3+k}, \quad (k=1,2), \\ -(dp)_{(\bar{\rho},\bar{\theta})}(\tilde{\rho},\tilde{\theta}) + \bar{\mu}'(\nabla\cdot\tilde{v}) + 2\bar{\mu}\dfrac{\partial\tilde{v}_3}{\partial y_3} - \sigma\nabla'^2F - \bar{p}_0'F\Big|_{y_3=0} = f^6, \\ \bar{\kappa}\,\dfrac{\partial\tilde{\theta}}{\partial y_3} + \kappa_e\tilde{\theta}\Big|_{y_3=0} = \kappa_e\tilde{\theta}_e + f^7, \quad on\ \boldsymbol{R}_T^2, \\ \tilde{v} = 0, \quad \tilde{\theta} = \theta_a, \quad on\ \Sigma_T, \\ F_t - \tilde{v}_3|_{y_3=0} = f^8, \quad on\ \boldsymbol{R}_T^2, \quad F|_{t=0} = F_0(y'), \quad on\ \boldsymbol{R}^2, \end{cases}$$

where $\tilde{\boldsymbol{V}} = \bar{\mu}'(\nabla\cdot\tilde{v})\boldsymbol{I} + 2\bar{\mu}\boldsymbol{D}(\tilde{v})$, $\bar{p}_0' = \frac{\partial}{\partial x_3}p(\bar{\rho}(x_3),\bar{\theta})\Big|_{x_3=0}$ and $f = \{f^i\ (i = 1,\dots,8)\}$ is at least quadratic functions of $(\tilde{\rho},\tilde{v},\tilde{\theta},\tilde{F})$ and their first and second derivatives. The estimate of the linearized problem (3.1),(3.2) with given f reads as follows.

Lemma 3.1 *Let* $b\in W_2^{3/2+l}$ *with* $l\in(1/2,1)$, $\tilde{\rho}_0,\tilde{v}_0,\tilde{\theta}_0\in W_2^{1+l}(\bar{\Omega})$, $F_0\in W_2^{5/2+l}(\boldsymbol{R}^2)$, $f^1\in W_2^{1+l,1/2+l/2}(\bar{Q}_T)$, $f^2,f^3\in W_2^{l,l/2}(\bar{Q}_T)$, $f^{3+k},f^6,f^7\in W_2^{1/2+l,1/4+l/2}(\boldsymbol{R}_T^2)$, $f^8\in W_2^{3/2+l,3/4+l/2}(\boldsymbol{R}_T^2)$, $\theta_e\in W_2^{3+l,3/2+l/2}(\boldsymbol{R}_T^3)$, $\theta_a\in W_2^{3/2+l,3/4+l/2}(\Sigma_T)$. *Then for the problem (3.1),(3.2), we have the estimate*

$$
\begin{aligned}
&\|\tilde{\rho}\|_{W_2^{1+l,1/2+l/2}(\bar{Q}_T)} + \|(\tilde{v},\tilde{\theta})\|_{W_2^{2+l,1+l/2}(\bar{Q}_T)} + \|F\|_{W_2^{5/2+l,5/4+l/2}(R_T^2)} \leq \\
&\leq c_6 \Big(\|(\tilde{\rho}_0,\tilde{v}_0,\tilde{\theta}_0)\|_{W_2^{1+l}(\bar{\Omega})} + \|F_0\|_{W_2^{5/2+l}(R^2)} + \\
&\quad + \|f^1\|_{W_2^{1+l,1/2+l/2}(\bar{Q}_T)} + \|(f^2,f^3)\|_{W_2^{l,l/2}(\bar{Q}_T)} + \\
&\quad + \|(f^{3+k},f^6,f^7)\|_{W_2^{1/2+l,1/4+l/2}(R_T^2)} + \|f^8\|_{W_2^{3/2+l,3/4+l/2}(R_T^2)} + \\
&\quad + \|\theta_e\|_{W_2^{3+l,3/2+l/2}(R_T^3)} + \|\theta_a\|_{W_2^{3/2+l,3/4+l/2}(\Sigma_T)} \Big).
\end{aligned}
\tag{3.3}
$$

We can prove Lemma3.1 by similar argument as in [4].
Let us proceed to the proof of Theorem2.1. First of all, estimating the norms of f in the right hand side of (3.3), we have

$$
\tilde{E}^{2+l}(\bar{Q}_T) \leq c_7 \left(E_{0,T} + \delta_1 \tilde{E}^{2+l}(\bar{Q}_T) + (\tilde{E}^{2+l}(\bar{Q}_T))^2 \right)
$$

which implies

$$
\tilde{E}^{2+l}(\bar{Q}_T) \leq 2c_7 E_{0,T} \tag{3.4}
$$

provided that the numbers ε_1 and δ_1 are small enough $2c_7\delta_1 + 4c_7^2\varepsilon_1 < 1$.
Next we rewrite the problem for $\tilde{\theta}$ as

$$
\begin{cases}
\bar{\rho}\bar{c}_V\tilde{\theta}_t - \bar{\kappa}\nabla^2\tilde{\theta} = f^3 - \bar{\theta}\bar{p}_\rho(\nabla\cdot\tilde{v}) - \\
\qquad\qquad -\bar{\kappa}\nabla^2\tilde{\theta} + \nabla\cdot(\bar{\kappa}\nabla\tilde{\theta}) \equiv f'^3, \quad in\ \bar{Q}_T, \\
\tilde{\theta}|_{t=0} = \tilde{\theta}_0(y), \quad on\ \bar{\Omega}, \\
\bar{\kappa}\left.\dfrac{\partial\tilde{\theta}}{\partial y_3}\right|_{y_3=0} = \kappa_e(\tilde{\theta}_e - \tilde{\theta}) + f^7 \equiv f'^7, \quad on\ \boldsymbol{R}_T^2, \\
\tilde{\theta} = \theta_a, \quad on\ \Sigma_T,
\end{cases}
$$

and apply the well-known estimate for the heat equation to obtain

$$\begin{aligned}\|\tilde{\theta}\|_{W_2^{3+l,3/2+l/2}(\bar{Q}_T)} \le & c_8 \left(\|\tilde{\theta}_0\|_{W_2^{2+l}(\bar{\Omega})} + \|f'^3\|_{W_2^{1+l,1/2+l/2}(\bar{Q}_T)} + \right. \\ & \left. + \|f'^7\|_{W_2^{3/2+l,3/4+l/2}(R_T^2)} + \|\theta_a\|_{W_2^{5/2+l,5/4+l/2}(\Sigma_T)} \right) \\ \le & c_9 \left(E_{0,T} + \delta_1 \|\tilde{\theta}\|_{W_2^{3+l,3/2+l/2}(\bar{Q}_T)} \right),\end{aligned}$$

here, of course, we have used (3.4). Hence the estimate

$$\|\tilde{\theta}\|_{W_2^{3+l,3/2+l/2}(\bar{Q}_T)} \le 2c_9 E_{0,T} \tag{3.5}$$

follows provided $c_9\delta_1 < \frac{1}{2}$.
Finally, for the estimate of highest derivatives of $(\tilde{\rho}, \tilde{v}, F)$, we appeal to the energy method. The idea is similar to that of Matsumura and Nishida ([3]) but we use finite differences since we work our problem in fractional power spaces ([6]). It is convenient for a moment to rewrite the problem for $(\tilde{\rho}, \tilde{v}, F)$ as

$$\left\{\begin{array}{ll} \mathcal{L}^1(\tilde{\rho}, \tilde{v}) \equiv \dfrac{\tilde{D}\tilde{\rho}}{Dt} + \bar{\rho}(\tilde{\nabla} \cdot \tilde{v}) = g^1, & \\ \mathcal{L}^2(\tilde{\rho}, \tilde{v}) \equiv \bar{\rho}\tilde{v}_t - \tilde{\nabla} \cdot \bar{\boldsymbol{P}}(\tilde{\rho}, \tilde{v}) = g^2, & in\ \bar{Q}_T, \\ \bar{\boldsymbol{P}}(\tilde{\rho}, \tilde{v}) e_3 - \sigma \nabla'^2 F e_3\big|_{y_3=0} = g^3, & \tilde{v}|_\Sigma = 0, \\ F_t - \tilde{v}_3|_{y_3=0} = g^4 & \end{array}\right. \tag{3.6}$$

where $\frac{\tilde{D}}{Dt} = \frac{\partial}{\partial t} - (B \cdot \tilde{\nabla}) + (\tilde{v} \cdot \tilde{\nabla})$, $\tilde{\nabla} = \tilde{A}\nabla_y$, $\tilde{A} =^t (\frac{\partial x_i}{\partial y_j})^{-1}{}_{1 \le i,j \le 3}$, $B = (\frac{\partial x_i}{\partial t})_{1 \le i \le 3}$, $\bar{\boldsymbol{P}}(\tilde{\rho}, \tilde{v}) = (-\bar{p}_\rho \tilde{\rho} + \bar{\mu}'(\tilde{\nabla} \cdot \tilde{v}))\boldsymbol{I} + 2\bar{\mu}\boldsymbol{D}(\tilde{v})$ and here and in what follows, the terms $g^i (i = 1, 2, \cdots)$ being thus defined. We shall begin with the estimates of the derivatives with respect to t. Let us put

$$\Delta_t^k(h)\tilde{f}(y,t) = \sum_{j=0}^{k} C_k^j (-1)^{k-j} \tilde{f}(y, t + jh),$$

$k > \frac{1}{2}(1 + l)$, $C_k^j = \binom{k}{j}$ and let $\varphi(t)$ be a smooth function vanishing for $t \le t_0$ and equals to 1 for $t \ge 2t_0$.

Lemma 3.2 *For $(\tilde{\rho}, \tilde{v}, F)$ the inequalities*

$$
(3.7) \quad \begin{aligned}
&\int_0^{h_0} \frac{dh}{h^{2+l}} \left[\varphi(t) \int_{\bar{\Omega}} (|\Delta_t^k(h)\tilde{\rho}|^2 + |\Delta_t^k(h)\tilde{v}|^2) dy\right]_{T-kh_0} \leq \\
&\leq c_{10} \left(E_{0,T}^2 + \int_0^{h_0} \frac{dh}{h^{2+l}} \int_0^{T-kh_0} \varphi(t)(|S_0| + |G_0|) dt\right),
\end{aligned}
$$

$$
(3.8) \quad \begin{aligned}
&\int_0^{h_0} \frac{dh}{h^{2+l}} \int_0^{T-kh_0} \varphi(t) dt \int_{\bar{\Omega}} (|\Delta_t^k(h)\tilde{\rho}_t|^2 + |\Delta_t^k(h)\tilde{v}_t|^2) dy + \\
&\quad + \int_0^{h_0} \frac{dh}{h^{2+l}} \int_0^{T-kh_0} \varphi(t) dt \int_{R^2} (|\Delta_t^{k+1/4}(h) F_t|^2 dy' \leq \\
&\leq c_{11} \left(E_{0,T}^2 + \int_0^{h_0} \frac{dh}{h^{2+l}} \left[\varphi(t) \int_{\bar{\Omega}} |\Delta_t^k(h)\tilde{\rho}|^2 dy\right]_{T-kh_0} + \right. \\
&\quad \left. + \int_0^{h_0} \frac{dh}{h^{2+l}} \int_0^{T-kh_0} \varphi(t)(|S_1| + |G_1| + |G_1'|) dt\right),
\end{aligned}
$$

hold true, where

$$
S_i = \int_{R^2} \Delta_t^k(h) \partial_t^i \tilde{v} \cdot \bar{\boldsymbol{P}}(\Delta_t^k(h)\tilde{\rho}, \Delta_t^k(h)\tilde{v}) e_3 dy',
$$

$$
G_i = \int_{\bar{\Omega}} \left(\frac{\bar{p}_\rho}{\bar{\rho}} \Delta_t^k(h) \partial_t^i \tilde{\rho} \cdot \Delta_t^k(h) g'^1 + \Delta_t^k(h) \partial_t^i \tilde{v} \cdot \Delta_t^k(h) g^2\right) dy, \quad (i = 0, 1),
$$

$$
G_1' = \int_{R^2} \Delta_t^{k+1/4}(h) F_t \cdot \Delta_t^{k+1/4}(h) g^4 dy', \quad g'^1 = g^1 - \frac{\tilde{D}\tilde{\rho}}{Dt}
$$

and we have assumed $T > max\{kh_0, 2t_0\}$.

proof The identities

$$
\frac{\bar{p}_\rho}{\bar{\rho}} \Delta_t^k(h)\tilde{\rho} \cdot \Delta_t^k(h)(\mathcal{L}^1 - g^1) + \Delta_t^k(h)\tilde{v} \cdot \Delta_t^k(h)(\mathcal{L}^2 - g^2) = 0,
$$

$$
\begin{aligned}
&\Delta_t^k(h)\tilde{\rho}_t \cdot \Delta_t^k(h)(\mathcal{L}^1 - g^1) + \Delta_t^k(h)\tilde{v}_t \cdot \Delta_t^k(h)(\mathcal{L}^2 - g^2) + \\
&\quad + \Delta_t^{k+1/4}(h) F_t \cdot \Delta_t^{k+1/4}(h)(F_t - \tilde{v} - g^4) = 0
\end{aligned}
$$

yield the estimates (3.7),(3.8) respectively by integration by parts. This completes the proof of Lemma 3.2. ∎

The estimates of the derivatives with respect to y are derived from local considerations. We only consider here near the upper surface, since the case of the interior domain or near the lower bottom are easier. We introduce local rectangular coordinate system with the origin at some point $y^{(k)} = (y'^{(k)}, 0) \in \bar{\Gamma} \equiv \{y_3 = 0\}$ in a parallel direction with $\{y\}$ axes and consider the subdomains

$$\bar{\omega}^{(k)} = \{|y' - y'^{(k)}| \leq d,\ -2d \leq y_3 \leq 0\},$$
$$\bar{\Omega}^{(k)} = \{|y' - y'^{(k)}| \leq 2d,\ -4d \leq y_3 \leq 0\} \quad (d > 0)$$

and the associated smooth functions $\zeta^{(k)} \in C_0^\infty(\boldsymbol{R}^3)$ such that $\zeta^{(k)}(y) = 1$ if $y \in \omega^{(k)}$, $= 0$ if $y \in \bar{\Omega} - \bar{\Omega}^{(k)}$ and $0 \leq \zeta^{(k)} \leq 1$. The similar argument as Lemma 3.2 yields the estimate of the differences to tangential direction

$$\Delta^s(z')\tilde{f}(y,t) = \sum_{k=0}^{s} C_s^k (-1)^{s-k} \tilde{f}(y' + kz', y_3, t) \quad (s > 2 + l).$$

Lemma 3.3 *For any positive number ε_2 it holds that*

$$\begin{aligned}
&\int_{|z| \leq \frac{d}{s}} \frac{dz}{z^{3+2l}} \int_{\bar{\Omega}^{(k)}} (|\Delta^s(z')\tilde{\rho}_t|^2 + |\Delta^s(z')\tilde{v}_t|^2)\zeta^{(k)2} dy + \\
&\quad + \int_0^t \int_{|z| \leq \frac{d}{s}} \frac{dz}{z^{3+2l}} \int_{\bar{\Omega}^{(k)}} (|\nabla \Delta^s(z')\tilde{v}|^2 + |\Delta^s(z')\frac{\tilde{D}\tilde{\rho}}{Dt}|^2)\zeta^{(k)2} dy \leq \\
(3.9) \quad &\leq c_{12} \left(E_{0,T}^2 + \varepsilon_2 \int_0^t dt \int_{|z| \leq \frac{d}{s}} \frac{dz}{z^{3+2l}} \int_{\bar{\Omega}^{(k)}} |\Delta^s(z')\tilde{\rho}|^2 dy + \right. \\
&\quad \left. + \int_0^t \int_{|z| \leq \frac{d}{s}} \frac{(|S_2| + |G_2|)}{z^{3+2l}} dz \right),
\end{aligned}$$

where

$$S_2 = \int_{R^2} \Delta^s(z')\tilde{v} \cdot \bar{\boldsymbol{P}}(\Delta^s(z')\tilde{\rho}, \Delta^s(z')\tilde{v}) e_3 \zeta^{(k)2} dy',$$
$$G_2 = \int_{\bar{\Omega}^{(k)}} \left(\frac{\bar{p}_\rho}{\bar{\rho}} \Delta^s(z')\tilde{\rho} \cdot \Delta^s(z') g^{1'} + \Delta^s(z')\tilde{v} \cdot \Delta^s(z') g^2 \right) \zeta^{(k)2} dy.$$

We proceed to estimate the differences to normal direction in the line with [3]. This time we rewrite the equation $(3.6)_2$ in the form

$$\bar{\rho}\tilde{v}_t - \bar{\mu}\nabla^2\tilde{v} - (\bar{\mu}+\bar{\mu}')\nabla(\nabla\cdot\tilde{v}) + \bar{p}_\rho\nabla\tilde{\rho} = g^5. \tag{3.10}$$

If we eliminate $\tilde{v}_{3,y_3y_3}$ from the third component of (3.10) and

$$(\frac{\tilde{D}\tilde{\rho}}{Dt})_{y_3} + \bar{\rho}(\nabla\cdot\tilde{v})_{y_3} = g^1_{y_3} - [\nabla_{y_3},\bar{\rho}]\nabla\cdot\tilde{v} \equiv g^6,$$

we have

$$\begin{aligned}&\frac{(2\bar{\mu}+\bar{\mu}')}{\bar{\rho}}(\frac{\tilde{D}\tilde{\rho}}{Dt})_{y_3} + \bar{p}_\rho\tilde{\rho}_{y_3} = -\bar{\rho}\tilde{v}_{3,t} + \frac{(2\bar{\mu}+\bar{\mu}')}{\bar{\rho}}g^6 + \\ &+\bar{\mu}(\tilde{v}_{3,y_1y_1}+\tilde{v}_{3,y_2y_2}) - \bar{\mu}(\tilde{v}_{1,y_1}+\tilde{v}_{2,y_2})_{y_3} + g^5_3 \equiv g^7.\end{aligned}$$

Further, operating $\Delta^k(z')\Delta^m(z_3)$ yields

$$\begin{aligned}&\frac{(2\bar{\mu}+\bar{\mu}')}{\bar{\rho}}\Delta^k(z')\Delta^m(z_3)(\frac{\tilde{D}\tilde{\rho}}{Dt})_{y_3} + \bar{p}_\rho\Delta^k(z')\Delta^m(z_3)\tilde{\rho}_{y_3} = \\ &= \Delta^k(z')\Delta^m(z_3)g^7 - \Delta^k(z')[\Delta^m(z_3),\frac{(2\bar{\mu}+\bar{\mu}')}{\bar{\rho}}](\frac{\tilde{D}\tilde{\rho}}{Dt})_{y_3} - \\ &-\Delta^k(z')[\Delta^m(z_3),\bar{p}_\rho]\tilde{\rho}_{y_3} \equiv g^8.\end{aligned} \tag{3.11}$$

Multiplying (3.11) by $\Delta^k(z')\Delta^m(z_3)\tilde{\rho}_{y_3}$ and $\Delta^k(z')\Delta^m(z_3)(\frac{\tilde{D}\tilde{\rho}}{Dt})_{y_3}$ and adding them, we have

Lemma 3.4

$$\begin{aligned}&\int_{|z|\le\frac{d}{s}}\frac{dz}{z^{3+2l}}\int_{\tilde{\Omega}^{(k)}}(|\Delta^s(z')\Delta^m(z_3)\tilde{\rho}_{y_3}|^2\zeta^{(k)2}dy + \\ &+\int_0^t\int_{|z|\le\frac{d}{s}}\frac{dz}{z^{3+2l}}\int_{\tilde{\Omega}^{(k)}}\Big(|\Delta^s(z')\Delta^m(z_3)\tilde{\rho}_{y_3}|^2 + \\ &+|\Delta^s(z')\Delta^m(z_3)(\frac{\tilde{D}\tilde{\rho}}{Dt})_{y_3}|^2\Big)\zeta^{(k)2}dy \le \\ &\le c_{13}\left(E^2_{0,T} + \int_0^t\int_{|z|\le\frac{d}{s}}\frac{(|G_3|+|G_3'|+|G_4|)}{z^{3+2l}}dz\right),\end{aligned} \tag{3.12}$$

where

$$G_3 = \int_{\bar\Omega^{(k)}} \Delta^s(z')\Delta^m(z_3)\tilde\rho_{y_3} \cdot g^8\zeta^{(k)2}dy,$$

$$G_3' = \int_{\bar\Omega^{(k)}} \Delta^s(z')\Delta^m(z_3)(\frac{\tilde D\tilde\rho}{Dt})_{y_3} \cdot g^8\zeta^{(k)2}dy,$$

$$G_4 = \int_{\bar\Omega^{(k)}} \Delta^s(z')\Delta^m(z_3)\left((\frac{\tilde D\tilde\rho}{Dt})_{y_3} - \tilde\rho_{ty_3}\right) \cdot \Delta^s(z')\Delta^m(z_3)\tilde\rho_{y_3}\zeta^{(k)2}dy.$$

Finally, we consider incompressible Stokes system for $(u, q, \eta) \equiv \zeta^{(k)}$ $\Delta^k(z')(\tilde\rho, \tilde v, F)$, which reduces to the form

$$\text{(3.13)} \qquad \begin{cases} \nabla \cdot u = g^9 \equiv \nabla \cdot g'^9, \\ \bar\rho u_t - \bar\mu \nabla^2 u + \bar p_\rho \nabla q = g^{10}, \quad \text{in} \bar Q_T^{(k)} \equiv \bar\Omega^{(k)} \times (0,T), \\ u|_{t=0} \equiv u_o, \\ -\bar p_\rho \nabla q \boldsymbol{I} e_3 + 2\bar\mu \boldsymbol{D}(u)e_3 - \sigma \nabla'^2\eta|_{y_3=0} = g^{11}, \\ \eta_t - u_3 = g^{12}. \end{cases}$$

Applying the estimate analogous to (3.3), we obtain

$$\begin{aligned} &\|\zeta^{(k)}\Delta^k(z')\nabla\tilde\rho\|_{W_2^{m,m/2}(\bar Q_T^{(k)})} + \|\zeta^{(k)}\Delta^k(z')\tilde v\|_{W_2^{2+m,1+m/2}(\bar Q_T^{(k)})} + \\ &\quad + \|\zeta^{(k)}\Delta^k(z')F\|_{W_2^{5/2+m,5/4+m/2}(R_T^2)} \le \\ \text{(3.14)} \quad &\le c_{14}\Big(\|\zeta^{(k)}\Delta^k(z')\tilde v_0\|_{W_2^{1+m}(\bar\Omega^{(k)})} + \|\zeta^{(k)}\Delta^k(z')F_0\|_{W_2^{5/2+m}(R^2)} + \\ &\quad + \|g^9\|_{W_2^{1+m,1/2+m/2}(\bar Q_T^{(k)})} + \|g'^9\|_{W_2^{0,1+m/2}(\bar Q_T^{(k)})} + \|g^{10}\|_{W_2^{m,m/2}(\bar Q_T^{(k)})} + \\ &\quad + \|g^{11}\|_{W_2^{1/2+m,1/4+m/2}(R_T^2)} + \|g^{12}\|_{W_2^{3/2+m,3/4+m/2}(R_T^2)}\Big). \end{aligned}$$

From (3.4),(3.5) and Lemmas 3.2-3.5 together with some lengthy calculations connected with the terms in the right hand side of (3.7)-(3.14), we arrive (2.2). This completes the proof of Theorem 2.1. ∎

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